# Solutions for Exercise 3

1. We are asked to find
$$M_X(t) = E\left[e^{tX}\right]$$
for $X \sim N(\mu, \sigma^2)$. We may write $X = \mu + \sigma Z$ where $Z \sim N(0,1)$, so that
$$\begin{aligned} M_X(t) &= E\left[e^{t(\mu+\sigma Z)}\right] = e^{\mu t} E\left[e^{\sigma t Z}\right] \\ &= e^{\mu t} M_Z(\sigma t). \end{aligned}$$
So one need only find the MGF for a standard normal random variable.
$$\begin{aligned} M_Z(t) &= \int_{-\infty}^{\infty} e^{zt} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}[(z-t)^2 - t^2]} dz \\ &= e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} dz \\ &= e^{\frac{1}{2}t^2}. \end{aligned}$$
The last integral is the total probability for a $N(t,1)$ random variable, and so is equal to 1. The step from line 1 to line 2 follows from the simple algebraic identity
$$-\frac{1}{2}[(z-t)^2 - t^2] = -\frac{1}{2}z^2 + zt.$$
The MGF for the general normal distribution is
$$M_X(t) = \exp(\mu t + \sigma^2 t^2/2).$$
The mean of $Y$ is
$$E[e^X] = M_X(1) = \exp(\mu + \sigma^2/2).$$
Also,
$$E[Y^2] = M_X(2) = \exp(2\mu + 4\sigma^2/2).$$
The variance of $Y$ is therefore
$$\exp(2\mu + 4\sigma^2/2) - \exp(2\mu + 2\sigma^2/2) = \exp(2(\mu + \sigma^2/2))(\exp(\sigma^2) - 1).$$

2. The Markov inequality says that if $X$ is a non-negative random variable, then for $\lambda > 0$,
$$P[X \geq \lambda] \leq \frac{E[X]}{\lambda}.$$
The proof is fairly easy.
$$\begin{aligned} P[X \geq \lambda] &= E[I_{[\lambda,\infty)}(X)] \\ &\leq E[\frac{X}{\lambda} I_{[\lambda,\infty)}(X)] \\ &= \frac{1}{\lambda} E[X I_{[\lambda,\infty)}(X)] \\ &\leq \frac{1}{\lambda} E[X]. \end{aligned}$$

 

If $t > 0$, then the events $Y \geq \gamma$ and $e^{tY} \geq e^{t\gamma}$ are the same. We can take, in the Markov inequality, $X = e^{tY}$ where $t > 0$, to obtain

$$\begin{aligned} P[Y \geq \gamma] = P[e^{tY} \geq e^{t\gamma}] &\leq \frac{E[e^{tY}]}{e^{t\gamma}} \\ &= \frac{M_Y(t)}{e^{t\gamma}}. \end{aligned}$$

Notice that this is true for every $t > 0$, so we can choose that value of $t$ that gives the sharpest upper bound.

If $Z \sim N(0,1)$, then $M_Z(t) = e^{t^2/2}$, so the previous inequality gives

$$P[Z \geq \gamma] \leq \frac{e^{t^2/2}}{e^{\gamma t}}$$

and choosing $t = \gamma$ gives, for $\gamma > 0$

$$P[Z \geq \gamma] \leq e^{-\gamma^2/2}.$$

The inequality is also obviously true for $\gamma = 0$. One can show that taking $t = \gamma$ gives the sharpest result.

3. (a) We know that

$$M_X(t) = \sum_{j=0}^{\infty} m_j \frac{t^j}{j!}$$

This gives

$$K_X(t) = \log M_X(t) = \log \left[ \sum_{j=0}^{\infty} m_j \frac{t^j}{j!} \right].$$

the right hand side is

$$\log \left[ 1 + m_1 \frac{t}{1!} + m_2 \frac{t^2}{2!} + m_3 \frac{t^3}{3!} + \dots \right]$$

Using the Taylor series expansion $\log(1+x) = x - x^2/2 + x^3/3 - \dots$, we get

$$\begin{aligned} K_X(t) = &\left[ m_1 \frac{t}{1!} + m_2 \frac{t^2}{2!} + m_3 \frac{t^3}{3!} + \dots \right] \\ &- \left[ m_1 \frac{t}{1!} + m_2 \frac{t^2}{2!} + m_3 \frac{t^3}{3!} + \dots \right]^2 \Big/ 2 \\ &+ \left[ m_1 \frac{t}{1!} + m_2 \frac{t^2}{2!} + m_3 \frac{t^3}{3!} + \dots \right]^3 \Big/ 3 + \dots \end{aligned}$$

Retaining terms up to degree $t^3$ gives

$$\begin{aligned} K_X(t) = &\left[ m_1 t/1! + m_2 t^2/2! + m_3 t^3/3! + \dots \right] \\ &- \left[ (m_1)^2 t^2 + 2 m_1 m_2 t^3/2 + \dots \right]/2 \\ &+ \left[ (m_1)^3 t^3 \right]/3 + \dots. \end{aligned}$$

This gives

$$\begin{aligned} K_X(t) = &m_1 \frac{t}{1!} + \left[ m_2 - (m_1)^2 \right] t^2/2! \\ &+ \left[ m_3 - 3 m_1 m_2 + 2 (m_1)^3 \right] t^3/3! + \dots. \end{aligned}$$

So, identifying coefficients of $t^r/r!$ on each side for $r = 1, 2, 3$

$$\begin{aligned}
\kappa_1 &= m_1 \\
\kappa_2 &= m_2 - (m_1)^2 = \mu_2 \\
\kappa_3 &= m_3 - 3m_1m_2 + 2(m_1)^3 = \mu_3.
\end{aligned}$$

(b) IF $Y = X - m_1$, then by definition, $m_r^{(Y)} = \mu_r^{(X)} = \mu_r$ for $r = 2, 3, \ldots$. We can also see that

$$\begin{aligned}
M_Y(t) &= E[e^{Yt}] \\
&= E[e^{t(X-m_1)}] \\
&= e^{-tm_1} M_X(t).
\end{aligned}$$

It follows that

$$\begin{aligned}
K_Y(t) &= -tm_1 + K_X(t) \\
&= -\kappa_1 t + K_X(t).
\end{aligned}$$

It follows that **all** the cumulants of $Y$ and $X$ must be the same except for the first. In other words, the cumulants of order greater than 1 **do not depend on the location** of the distribution. We can get $\kappa_2, \kappa_3, \kappa_4, \kappa_5$ from the moments of $Y$, which are just the central moments for $X$.

$$K_Y(t) = \log[1 + \mu_2 t^2/2! + \mu_3 t^3/3! + \mu_4 t^4/4! + \mu_5 t^5/5! + \ldots]$$

Once again expanding the right hand side we get

$$\begin{aligned}
K_Y(t) = [\mu_2 t^2/2! + \mu_3 t^3/3! + \mu_4 t^4/4! + \mu_5 t^5/5! + \ldots] \\
- [\mu_2 t^2/2! + \mu_3 t^3/3! + \mu_4 t^4/4! + \mu_5 t^5/5! + \ldots]^2/2 + \ldots
\end{aligned}$$

Keeping terms to the fifth power of t,

$$\begin{aligned}
K_Y(t) = [\mu_2 t^2/2! + \mu_3 t^3/3! + \mu_4 t^4/4! + \mu_5 t^5/5! + \ldots] \\
- [\mu_2^2 t^4/4 + 2\mu_2\mu_3 t^5/12 + \ldots]/2 + \ldots
\end{aligned}$$

which gives

$$\begin{aligned}
K_Y(t) = \mu_2 t^2/2! + \mu_3 t^3/3! + (\mu_4 - 3\mu_2^2)t^4/4! \\
+ (\mu_5 - 10\mu_3\mu_2)t^5/5! + \ldots.
\end{aligned}$$

Picking out the coefficients of $t^r/r!$ on each side

$$\begin{aligned}
\kappa_2^{(Y)} &= \mu_2 \\
\kappa_3^{(Y)} &= \mu_3 \\
\kappa_4^{(Y)} &= \mu_4 - 3\mu_2^2 \\
\kappa_5^{(Y)} &= \mu_5 - 10\mu_3\mu_2.
\end{aligned}$$

 

(c) If $Z = Y/\sigma$, then

$$K_Z(t) = \log M_Z(t) = \log E[e^{Zt}] = \log E[e^{Yt/\sigma}] = K_Y(t/\sigma).$$

It follows that $\kappa_r^{(Z)} = \kappa_r^{(Y)}/\sigma^r$, and so

$$\begin{aligned}
\kappa_2^{(Z)} &= 1 \\
\kappa_3^{(Z)} &= \mu_3/\mu_2^{3/2} \\
\kappa_4^{(Z)} &= (\mu_4 - 3\mu_2^2)/\mu_2^2 \\
\kappa_5^{(Z)} &= (\mu_5 - 10\mu_3\mu_2)/\mu_2^{5/2}.
\end{aligned}$$

(d) For the normal distribution, $K_X(t) = \mu t + \sigma^2 t^2/2$, so that

$$\begin{aligned}
\kappa_1 &= \mu \\
\kappa_2 &= \sigma^2 \\
\kappa_r &= 0 \text{ for } r > 2.
\end{aligned}$$

So

$$\begin{aligned}
m_1 &= \kappa_1 = \mu \\
m_2 &= \kappa_2 + (m_1)^2 = \sigma^2 + \mu^2 \\
m_3 &= \kappa_3 + 3m_2m_1 - 2(m_1)^3 = 3\sigma^2\mu + \mu^3.
\end{aligned}$$

4. We shall require that $-1 < t < 1$ for some parts of the following derivation:

$$\begin{aligned}
M_X(t) &= \int_{-\infty}^{\infty} e^{xt}\frac{e^{-|x|}}{2}dx \\
&= \int_{-\infty}^{0} e^{xt}\frac{e^{x}}{2}dx + \int_{0}^{\infty} e^{xt}\frac{e^{-x}}{2}dx \\
&= \int_{-\infty}^{0} \frac{e^{x(1+t)}}{2}dx + \int_{0}^{\infty} \frac{e^{-x(1-t)}}{2}dx \\
&= \frac{e^{x(1+t)}}{2(1+t)}\Big|_{-\infty}^{0} + \frac{-e^{-x(1-t)}}{2(1-t)}\Big|_{0}^{\infty} \\
&= \frac{1}{2(1+t)} + \frac{1}{2(1-t)} \\
&= \frac{1}{1-t^2}.
\end{aligned}$$

$$\begin{aligned}
K_X(t) &= -\log(1-t^2) = t^2 + (t^2)^2/2 + (t^2)^3/3 + \ldots \\
&= 0t + 2\frac{t^2}{2!} + 0\frac{t^3}{3!} + 12\frac{t^4}{4!} + \ldots.
\end{aligned}$$

So,

$$\begin{aligned}
\kappa_1 &= 0 \\
\kappa_2 &= 2 \\
\kappa_3 &= 0 \\
\kappa_4 &= 12.
\end{aligned}$$

 

5. (a) This can't work, because if it were true

$$F(\infty, \infty) = H(\infty) + G(\infty) = 1 + 1 = 2.$$

This is inconsistent with $F(\infty, \infty)$ being a probability.

(b) If we consider the joint distribution of independent random variables $X$ with distribution function $H(x)$ and $Y$ with distribution function $G(y)$, then

$$\begin{aligned} F(x, y) &= P[X \le x, Y \le y] \\ &= P[X \le x]P[Y \le y] \\ &= H(x)G(y). \end{aligned}$$

This verifies that $H(x)G(y)$ is a joint distribution function.

(c) This one does not work because

$$F(-\infty, \infty) = \max[H(-\infty), G(\infty)] = \max[0, 1] = 1.$$

But this should be 0.

(d) As the question suggests, consider the singular joint distribution for which $X$ and $Y$ always take exactly the same value, and for which the marginal distribution function of $X$ is $H(x)$. A typical value for $(X, Y)$ is $(x, x)$. The support of this distribution in the $XY$ plane is the line $y = x$. We have

$$\begin{aligned} F(x, y) &= P[X \le x, Y \le y] \\ &= P[X \le x, X \le y] \\ &= \min[H(x), H(y)]. \end{aligned}$$

So we have shown that this is a distribution function, though for a singular distribution.

A more difficult exercise would be to show that $F(x, y) = \min[H(x), G(y)]$ is a joint distribution function.

 